

HYPERCYCLICITY OF DERIVATIONS
AND
OPTIMAL GROWTH OF
FREQUENTLY HYPERCYCLIC
HARMONIC FUNCTIONS

Clifford Gilmore

Department of Mathematics and Statistics
Faculty of Science
University of Helsinki

Academic Dissertation

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Supervisor

Dr Hans-Olav Tylli, University of Helsinki, Finland

Preliminary Examiners

Professor Karl Grosse-Erdmann, Université de Mons, Belgium

Professor Richard Timoney, Trinity College Dublin, Ireland

Opponent

Professor José Bonet, Universitat Politècnica de València, Spain

Custos

Professor Eero Saksman, University of Helsinki, Finland

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List of Included Articles

This dissertation consists of an introductory part and the following research articles.

- [A] C. Gilmore, E. Saksman and H.-O. Tylli. Hypercyclicity properties of commutator maps. *Integr. Equ. Oper. Theory*, 87(1):139-155, 2017.
- [B] C. Gilmore. Dynamics of generalised derivations and elementary operators. 2016. ArXiv:1605.07409.
- [C] C. Gilmore, E. Saksman and H.-O. Tylli. Optimal growth of harmonic functions frequently hypercyclic for the partial differentiation operator. 2017. ArXiv:1708.08764.

Articles [A] and [C] contain significant contributions from the author.

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The gross and net result of it is that people who spent most of their natural lives riding iron bicycles over the rocky roadsteads of this parish get their personalities mixed up with the personalities of their bicycle as a result of the interchanging of the atoms of each of them and you would be surprised at the number of people in these parts who nearly are half people and half bicycles.

—Flann O’Brien, *The Third Policeman*

1. Introduction

The orbits of linear operators can be fantastically complicated and linear dynamics exhibits the same beauty and complexity as nonlinear dynamics. It has been known for sometime that continuous linear operators on Hilbert space can actually be chaotic! In fact, the orbits of linear operators can be as complicated as the orbits of any continuous function.

—Feldman [39]

Chaos is typically viewed as a nonlinear phenomenon. Indeed, the area of classical dynamical systems studies the mathematical rules governing the long-term evolution of nonlinear phenomena such as climate, turbulence, fluid dynamics and even economics. However, it is by now well established that seemingly tractable linear systems may give rise to linear chaos.

Chaos theory has been described in lay terms as the ‘science of surprises’ and in everyday usage chaos commonly depicts a state of disorder. So it is natural to ask, what mathematically precise definition captures the essential properties that causes a dynamical system to be chaotic?

By a dynamical system we mean the pair (X, T) , where X is a metric space and T is a continuous map acting on X . Devaney [35] suggested the dynamical system (X, T) is chaotic if it possesses the following characteristics:

- (1) long term unpredictability,
- (2) it cannot be simplified,
- (3) it has some regularity.

He proposed that these characteristics are captured by the following mathematical properties.

The first characteristic corresponds to the notion of *sensitive dependence on initial conditions*, which is frequently referred to as the *butterfly effect*. It is considered the essence of chaos and it describes the situation where small discrepancies in the initial state of the system may lead to vastly differing outcomes. It explains, for instance, the difficulty in obtaining accurate long-term weather forecasts.

The second characteristic is captured by topological transitivity. The continuous map $T: X \rightarrow X$ is said to be *topologically transitive* if for any pair of nonempty, open subsets $U, V \subset X$, there exists some $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. As illustrated in Figure 1.1, under the action of T every non-trivial part of X will eventually visit the whole space. Hence the system cannot be simplified or reduced into smaller and potentially more manageable components.

Devaney defined the last characteristic to be when the map T possesses a dense set of periodic points. We recall $x \in X$ is a periodic point for T if there exists $n \geq 1$ such that $T^n x = x$.

If the space X has a linear structure and $T: X \rightarrow X$ is a continuous linear map, then we say (X, T) is a linear dynamical system. The central notion

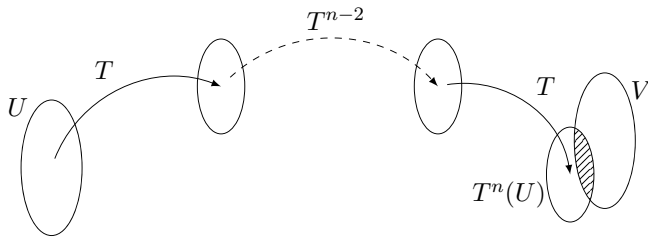


FIGURE 1.1. Topological transitivity

of linear dynamics is hypercyclicity, since it encompasses both topological transitivity and sensitive dependence on initial conditions. More precisely, in this setting hypercyclicity is equivalent to topological transitivity via the Birkhoff Transitivity Theorem and moreover hypercyclicity implies sensitive dependence on initial conditions. We do not elaborate further on the property of possessing a dense set of periodic points since it does not play any role in the sequel. In fact, some alternative definitions of chaos discard it completely.

This thesis is focused on the hypercyclic and frequently hypercyclic properties of particular classes of operators. In the remainder of this section we introduce these notions and we recall the pertinent aspects of linear dynamics.

1.1. Hypercyclicity

Unless otherwise stated, for the remainder of this section X is a separable Fréchet space and we denote the space of continuous linear operators on X by $\mathcal{L}(X)$.

We say $T \in \mathcal{L}(X)$ is *hypercyclic* if there exists $x \in X$ such that its T -orbit is dense in X , that is

$$\overline{\{T^n x : n \geq 0\}} = X.$$

Such an $x \in X$ is called a hypercyclic vector for T .

The first example of a continuous linear operator with a dense orbit was given by Birkhoff [18] in 1929 when he demonstrated that translation operators $f(x) \mapsto f(x + a)$ are hypercyclic on the space $H(\mathbb{C})$ of entire holomorphic functions on the complex plane \mathbb{C} , for $a \neq 0$. MacLane [60] subsequently proved in 1952 that the differentiation operator $D: f \mapsto f'$ is hypercyclic on $H(\mathbb{C})$.

The first instances of hypercyclic operators in the Banach space setting were identified in 1969 by Rolewicz [72]. For $X = c_0$, the space of sequences with limit equal to zero, or $X = \ell^p$, the space of p -summable sequences for $1 \leq p < \infty$, he proved that scalar multiples of the backward shift operator $cB \in \mathcal{L}(X)$ are hypercyclic when $|c| > 1$. We recall the unilateral backward shift $B \in \mathcal{L}(X)$ is defined as

$$B(x_1, x_2, \dots) = (x_2, x_3, \dots)$$

for $(x_n) \in X$.

Linear dynamics began to receive systematic attention in the late 1980s following the work of Kitai [56] and Gethner and Shapiro [45]. It subsequently developed into a considerable branch of operator theory and comprehensive accounts of the area can be found in [14] and [50].

One of the principal results of Kitai [56], which was independently re-discovered in [45], is a sufficient condition for hypercyclicity known as the Hypercyclicity Criterion. It has emerged as an important tool in linear dynamics since it is not always straightforward to identify an explicit hypercyclic vector for a particular hypercyclic operator. We present the standard version of the criterion below, which can be found in [14, Definition 1.5] or [50, Theorem 3.12].

We say $T \in \mathcal{L}(X)$ satisfies the Hypercyclicity Criterion if there exist dense subsets $X_0, Y_0 \subset X$, an increasing sequence (n_k) of positive integers and maps $S_{n_k}: Y_0 \rightarrow X$, $k \geq 1$, such that for any $x \in X_0$, $y \in Y_0$ one has

- (i) $T^{n_k}x \rightarrow 0$,
- (ii) $S_{n_k}(y) \rightarrow 0$,
- (iii) $T^{n_k}S_{n_k}(y) \rightarrow y$

as $k \rightarrow \infty$. If T satisfies the Hypercyclicity Criterion then it is hypercyclic (cf. [14, Theorem 1.6] or [50, Theorem 3.12]).

We note, however, that the Hypercyclicity Criterion is a stronger property than hypercyclicity. It was a long-standing open problem in linear dynamics, originally posed by Herrero [54], whether every hypercyclic operator satisfies the Hypercyclicity Criterion. The question was resolved in 2006 by de la Rosa and Read [34], who constructed a Banach space and a hypercyclic operator that does not satisfy the criterion. A family of examples in the setting of classical ℓ^p spaces, for $1 \leq p < \infty$, was subsequently identified by Bayart and Matheron [13].

One motivation for investigating hypercyclic operators originated with the invariant subspace problem and the study of cyclic operators. The operator $T \in \mathcal{L}(X)$ is *cyclic* if there exists $x \in X$ (said to be a cyclic vector for T) such that the linear span of its orbit under T is dense in X , that is

$$\overline{\text{span}}\{T^n x : n \geq 0\} = X.$$

The invariant subspace problem asks, given $T \in \mathcal{L}(X)$, does there always exist a non-trivial, closed T -invariant subspace $W \subset X$? The subspace W is T -invariant if $T(W) \subset W$ and it is said to be non-trivial if $W \neq \{0\}$ and $W \neq X$.

It holds that T does not possess a non-trivial closed invariant subspace if and only if every nonzero $x \in X$ is a cyclic vector for T . Enflo constructed the first counter example in 1976 and subsequently Read [70] identified that there exists an operator T , acting on the classical Banach space ℓ^1 , such that every nonzero $x \in \ell^1$ is cyclic for T .

This naturally led research activity to the invariant subset problem and the study of hypercyclic operators. In fact, Read [71] later showed that there exists $T \in \mathcal{L}(\ell^1)$ such that every nonzero $x \in \ell^1$ is a hypercyclic vector for T . This connection also motivated Beauzamy [15] to coin the term hypercyclic.

Between the classes of cyclic and hypercyclic operators lie the supercyclic operators. We say $T \in \mathcal{L}(X)$ is *supercyclic* if there exists $x \in X$ such that its projective T -orbit is dense in X , that is

$$\overline{\{\lambda T^n x : n \geq 0, \lambda \in \mathbb{C}\}} = X.$$

The hypercyclic operators are strictly contained in the class of supercyclic operators [14, Example 1.15]. Supercyclicity is in fact a slightly older notion than hypercyclicity and the term supercyclic was introduced by Hilden and Wallen [55].

On the other hand, interest in hypercyclicity also stems from the more general notion of universality. For topological spaces X and Y , the countable family $(T_n)_{n \in \mathbb{N}}$ of continuous maps $T_n: X \rightarrow Y$ is *universal* if there exists $x \in X$ such that

$$\overline{\{T_n(x) : n \in \mathbb{N}\}} = Y$$

and such an $x \in X$ is called a universal element for (T_n) . If we let $X = Y$ be a topological vector space and we take the sequence (T_n) to be the iterates of a single continuous linear operator, then it follows that hypercyclicity is a particular instance of universality.

The discovery of universal power series was credited to Fekete [65] in 1914. He showed there exists a formal real power series $\sum_{n=1}^{\infty} a_n x^n$ on $[-1, 1]$ with the property that for any continuous function $g: [-1, 1] \rightarrow \mathbb{R}$ with $g(0) = 0$, there exists an increasing sequence of positive integers (n_k) such that

$$\sum_{n=1}^{n_k} a_n x^n \rightarrow g(x)$$

uniformly as $k \rightarrow \infty$. The observation of Fekete can be further extended to a universal Taylor series on all of \mathbb{R} , cf. [49, Section 3a].

Many of the statements for hypercyclicity have analogues for this more general notion. Indeed, the previously mentioned hypercyclicity results for the translation and differentiation operators were originally proven for universality. However, some of the most powerful tools used to investigate hypercyclic operators, for instance the spectral techniques, are not available in the case of universality. The term universality was coined by Marcinkiewicz [62] and an extensive survey on universal families can be found in [49].

We note that hypercyclicity is a purely infinite-dimensional phenomenon, since linear operators cannot have a dense orbit in the finite dimensional setting. On the other hand, Ansari [5] and Bernal [16] have shown that every separable, infinite-dimensional Banach space supports a hypercyclic operator. This was later generalised to the Fréchet space setting by Bonet and Peris [22].

We also recall that the set of hypercyclic vectors of a hypercyclic operator $T \in \mathcal{L}(X)$ is a dense G_δ subset of X (cf. [14, Theorem 1.2]). This fact makes available the powerful techniques of the Baire category theorem when investigating hypercyclicity.

Finally, another valuable tool we will encounter is the *hypercyclic comparison principle*, which was formulated by Shapiro [80]. For topological spaces X and X_0 , a continuous map $T: X \rightarrow X$ is said to be a *quasi-factor* of the continuous map $T_0: X_0 \rightarrow X_0$ if there exists a continuous map

$\Psi: X_0 \rightarrow X$ with dense range such that $T\Psi = \Psi T_0$. That is, the following diagram commutes.

$$\begin{array}{ccc} X_0 & \xrightarrow{T_0} & X_0 \\ \Psi \downarrow & & \downarrow \Psi \\ X & \xrightarrow{T} & X \end{array}$$

When T_0 and T are linear operators and the map Ψ can be taken as linear, then we say T is a *linear quasi-factor* of T_0 . The hypercyclic comparison principle states that hypercyclicity is preserved by quasi-factors, while supercyclicity and satisfying the Hypercyclicity Criterion are preserved by linear quasi-factors (cf. [14, Section 1.1.1]).

1.2. Frequent Hypercyclicity

The operator $T \in \mathcal{L}(X)$ is *frequently hypercyclic* if there exists $x \in X$ such that for any nonempty open subset $U \subset X$

$$\liminf_{N \rightarrow \infty} \frac{\#\{n : T^n x \in U, 0 \leq n \leq N\}}{N} > 0.$$

Here $\#$ denotes the cardinality of the set. Such an $x \in X$ is a *frequently hypercyclic vector* for T and the definition states that the set of indices, such that the T -orbit of x visits any given neighbourhood of X , has positive lower density.

The notion of frequent hypercyclicity was introduced by Bayart and Grivaux in 2004 [11, 12] and it is a stronger property than hypercyclicity. While hypercyclicity requires the orbit visits each neighbourhood of X , frequent hypercyclicity introduces the qualitative element of how frequently the orbit visits each neighbourhood. It stems from the notion of ergodicity in measurable dynamics and this approach to linear dynamics was originally due to Rudnicki [74] and Flytzanis [44].

It turns out the classical hypercyclic operators mentioned in the previous section (translation, differentiation and scalar multiples of the backward shift) are also frequently hypercyclic. Indeed, there are many results for hypercyclicity which have analogues in the frequently hypercyclic case. For instance the Frequent Hypercyclicity Criterion, which was initially identified in [12] and subsequently stated in the below form by Bonilla and Grosse-Erdmann [23].

We say $T \in \mathcal{L}(X)$ satisfies the *Frequent Hypercyclicity Criterion* if there exists a dense subset $X_0 \subset X$ and a map $S: X_0 \rightarrow X_0$ such that for any $x \in X_0$

- (i) $\sum_{n=0}^{\infty} T^n x$ converges unconditionally,
- (ii) $\sum_{n=0}^{\infty} S^n x$ converges unconditionally,
- (iii) $TSx = x$.

We recall the series $\sum_{n=1}^{\infty} x_n$ in a Fréchet space is called *unconditionally convergent* if for any bijection $\pi: \mathbb{N} \rightarrow \mathbb{N}$ the series

$$\sum_{n=1}^{\infty} x_{\pi(n)}$$

converges.

If T satisfies the Frequent Hypercyclicity Criterion then it is frequently hypercyclic (cf. [50, Theorem 9.9] or [14, Theorem 6.18]). We note, however, that there exist frequently hypercyclic operators that do not satisfy the Frequent Hypercyclicity Criterion [50, p. 248].

On the other hand, the class of frequently hypercyclic operators is strictly contained in the class of hypercyclic operators. For instance, in [14, Example 6.17] they show that the weighted backward shift $B_w: \ell^2 \rightarrow \ell^2$, given by

$$B_w(x_1, x_2, \dots) = (w_2x_2, w_3x_3, \dots), \quad \text{for } w_n = \sqrt{\frac{n+1}{n}}$$

and $(x_n) \in \ell^2$, is hypercyclic but not frequently hypercyclic.

Furthermore, in contrast to the hypercyclic case there exist separable, infinite-dimensional Banach spaces that do not support frequently hypercyclic operators [50, Corollary 9.41]. Another significant difference is the Baire category theorem is not at our disposal, since in general the set of frequently hypercyclic vectors is of the first Baire category [14, Theorem 6.25].

2. Hypercyclicity of Derivations

The questions considered in this section relate to articles [A] and [B]. They primarily investigate the hypercyclic properties of commutator maps and generalised derivations acting on separable operator ideals of $\mathcal{L}(X)$, where X is a Banach space. These questions are contained in the broader problem concerning the dynamics of elementary operators and we begin by introducing this more general class of operators. Unless otherwise stated, in this section X denotes a Banach space.

The *elementary operator* $\mathcal{E}_{A,B}: \mathcal{L}(X) \rightarrow \mathcal{L}(X)$ is induced by fixed $A, B \in \mathcal{L}(X)^n$ and defined as

$$\mathcal{E}_{A,B}(S) = \sum_{j=1}^n A_j S B_j = \sum_{j=1}^n L_{A_j} R_{B_j}(S)$$

for any $S \in \mathcal{L}(X)$ and where $A = (A_1, \dots, A_n)$, $B = (B_1, \dots, B_n) \in \mathcal{L}(X)^n$ are n -tuples of bounded linear operators on X . Here the *left* and *right multiplication* operators

$$L_U, R_T: \mathcal{L}(X) \rightarrow \mathcal{L}(X)$$

are, respectively, given by

$$L_U(S) = US, \quad R_T(S) = ST$$

for fixed $U, T \in \mathcal{L}(X)$ and any $S \in \mathcal{L}(X)$.

The term elementary operator was introduced in 1959 by Lumer and Rosenblum [59], however their study originated with Sylvester [84] in the setting of matrix algebras. The operator case was subsequently considered by Dalecki [33] and Rosenblum [73]. Since the 1950s the properties of elementary operators have been extensively studied and aspects of the theory have been surveyed in [31], [43], [6], [77] and [32]. They have also recently found applications in areas such as soliton physics [28] and quantum information theory [85, p. 134].

An important class of elementary operators are the *commutator maps* $\Delta_A: \mathcal{L}(X) \rightarrow \mathcal{L}(X)$, which are defined as

$$\Delta_A(S) = AS - SA = L_A(S) - R_A(S)$$

for any $S \in \mathcal{L}(X)$ and a fixed $A \in \mathcal{L}(X)$. They have been investigated from various perspectives, including the deep work of Anderson [3] and Stampfli [82, 83], which contains many elegant results on their range, norm and spectrum. Wider interest in commutator maps also comes from their connection to the Heisenberg uncertainty principle in quantum mechanics.

The maps Δ_A are also known as inner derivations since they satisfy the Leibniz rule

$$\Delta_A(SU) = S\Delta_A(U) + \Delta_A(S)U$$

for all $S, U \in \mathcal{L}(X)$. We briefly recall that a *derivation* on a Banach algebra \mathcal{A} is a linear map $\Delta: \mathcal{A} \rightarrow \mathcal{A}$ which satisfies

$$\Delta(ab) = a\Delta(b) + \Delta(a)b$$

for all $a, b \in \mathcal{A}$. It is well known that every derivation on $\mathcal{L}(X)$ is of the form Δ_A for some $A \in \mathcal{L}(X)$ and hence every derivation on $\mathcal{L}(X)$ is inner.

Commutator maps are a special case of the class of *generalised derivations* $\tau_{A,B}: \mathcal{L}(X) \rightarrow \mathcal{L}(X)$, which are defined as

$$\tau_{A,B}(S) = AS - SB = L_A(S) - R_B(S)$$

for any $S \in \mathcal{L}(X)$ and fixed $A, B \in \mathcal{L}(X)$. Generalised derivations, also known as intertwining maps, were first studied by Rosenblum [73], Lumer and Rosenblum [59] and Anderson and Foias [4]. They have also been extensively surveyed in [17] and [77]. In the setting of operator ideals of $\mathcal{L}(X)$ they have been investigated by, amongst others, Fialkow [40, 41, 42] and Maher [61]. They have applications in, for instance, hyperinvariant subspace theory [69], spectral operators [68], Lyapunov's equation and stability analysis [58], [52].

Before considering hypercyclicity on spaces of operators, we must first overcome the obstacle that for classical Banach spaces X the space $\mathcal{L}(X)$ is non-separable under the operator norm topology. One solution is to consider weaker topologies under which $\mathcal{L}(X)$ is separable. Indeed hypercyclicity of the left and right multipliers has been investigated in this setting in [29], [30], [64], [63], [21], [67] and [51].

However, the approach we take is to consider separable ideals of the space $\mathcal{L}(X)$. We say $(J, \|\cdot\|_J)$ is a *Banach ideal* of $\mathcal{L}(X)$ if

- (i) $J \subset \mathcal{L}(X)$ is a linear subspace,
- (ii) the norm $\|\cdot\|_J$ is complete in J and $\|S\| \leq \|S\|_J$ for all $S \in J$,
- (iii) $BSA \in J$ and $\|BSA\|_J \leq \|B\| \|A\| \|S\|_J$ for $A, B \in \mathcal{L}(X)$ and $S \in J$,
- (iv) the rank one operators $x^* \otimes x \in J$ and $\|x^* \otimes x\|_J = \|x^*\| \|x\|$ for all $x^* \in X^*$ and $x \in X$.

We recall the *rank one* operator $x^* \otimes x: X \rightarrow X$ is defined as

$$(x^* \otimes x)(z) = x^*(z)x$$

for $x^* \in X^*$, $x \in X$ and any $z \in X$. The space $\mathcal{F}(X)$ of *finite rank* operators is defined as the linear span of the rank one operators.

Instances of separable Banach ideals include the space $(\mathcal{N}(X), \|\cdot\|_N)$ of nuclear operators, with the nuclear norm, when the dual X^* is separable and the space $\mathcal{K}(X)$ of compact operators, under the operator norm topology, when X possesses the approximation property and X^* is separable [75]. When X is a separable Hilbert space, the spaces $(C_p, \|\cdot\|_p)$ of Schatten p -class operators, with the Schatten norm, give classical instances of separable Banach ideals for $1 \leq p < \infty$.

In this setting, Bonet et al. [21] used tensor techniques developed by Martínez-Giménez and Peris [63] to characterise when the left and right multipliers are hypercyclic on the space $\mathcal{N}(X)$ of nuclear operators. They then applied the hypercyclic comparison principle to extend the result to the space $\mathcal{K}(X)$ of compact operators.

The pertinent results in [21], expressed using the terminology of Banach ideals, are as follows. For a separable Banach ideal $J \subset \mathcal{L}(X)$, which contains the finite rank operators as a dense subset, it holds that

- (M1) L_A is hypercyclic on J if and only if $A \in \mathcal{L}(X)$ satisfies the Hypercyclicity Criterion,
- (M2) R_B is hypercyclic on J if and only if the adjoint B^* satisfies the Hypercyclicity Criterion on the dual X^* .

We note that in light of subsequent results which identify hypercyclic operators that do not satisfy the Hypercyclicity Criterion ([34], [13]), it follows from (M1) and (M2) that there exist hypercyclic operators A and B^* such that L_A and R_B are not hypercyclic.

Furthermore, although not explicitly stated in [21] their results also yield sufficient conditions for the hypercyclicity of the two-sided multipliers $L_A R_B$, for $A, B \in \mathcal{L}(X)$. To see this identify $L_A R_B$ with its tensor representation $B^* \otimes A$, whence hypercyclicity follows directly from the sufficient conditions given in [63].

Bonilla and Grosse-Erdmann [26, Theorem 8] subsequently uncovered a sufficient condition for when the left multiplier is frequently hypercyclic on separable Banach ideals by using tensor techniques in the spirit of [63].

Following the work of Bonet et al. [21], it is natural to investigate the hypercyclicity of the more general commutator maps and generalised derivations. These are the questions addressed in articles [A] and [B] and described in the sequel.

2.1. Hypercyclicity of Commutator Maps

Before considering the hypercyclicity of the commutator map Δ_A , we note that its range $\text{ran}(\Delta_A)$ in $\mathcal{L}(X)$ is usually quite small. For instance, on non-separable spaces such as $\mathcal{L}(\ell^p)$ the quotient

$$(2.1) \quad \mathcal{L}(\ell^p) / \overline{\text{ran}}(\Delta_A)$$

is non-separable for any $A \in \mathcal{L}(\ell^p)$, for $1 < p < \infty$ [83], [76].

Furthermore, it is well known that the identity map $I_X \notin \text{ran}(\Delta_A)$ for any $A \in \mathcal{L}(X)$ when X is infinite-dimensional and Halmos [53, p. 129] observed that no operator from the class of thin operators

$$\{\lambda I_X + K : \lambda \in \mathbb{C}, K \in \mathcal{K}(X)\}$$

lies in the range of Δ_A in $\mathcal{L}(X)$.

On the other hand, by a celebrated example of Anderson [3], there exist operators $A \in \mathcal{L}(\ell^2)$ such that the identity I_{ℓ^2} lies in the closure of the range $\overline{\text{ran}}(\Delta_A)$ in $\mathcal{L}(X)$. Moreover, restrictions of commutator maps to separable Banach ideals behave quite differently from (2.1). For instance, Stampfli [83] identified a compact operator K on the Hilbert space H such that $\overline{\text{ran}}(\Delta_K) = \mathcal{K}(H)$. (However, we will see later that commutator maps induced by compact operators cannot be hypercyclic.) Furthermore, it is a classical result that the restricted map

$$\Delta_B: \mathcal{K}(\ell^2) \rightarrow \mathcal{K}(\ell^2)$$

induced by the backward shift $B \in \mathcal{L}(\ell^2)$ has dense range on the ideal $\mathcal{K}(\ell^2)$ of the compact operators on ℓ^2 .

Notice for $A \in \mathcal{L}(X)$ that the iterates of Δ_A are of the form

$$(L_A - R_A)^n = \sum_{j=0}^n (-1)^j \binom{n}{j} L_A^{n-j} R_A^j.$$

This formula rapidly becomes cumbersome to work with directly and it also partly explains why the hypercyclicity of commutator maps is quite a subtle question.

To understand the dynamical behaviour of Δ_A we therefore employ approaches involving spectral theory, techniques from functional analysis and some ad hoc arguments using complex analysis. In particular, there exists a wealth of spectral conditions in the theory that prove useful in our work. We combine these spectral results with the following elegant formula for the spectrum of the generalised derivation $\tau_{A,B}$ when restricted to the Banach ideal $J \subset \mathcal{L}(X)$

$$(2.2) \quad \sigma_J(\tau_{A,B}) = \sigma(A) - \sigma(B) = \{\lambda - \mu : \lambda \in \sigma(A), \mu \in \sigma(B)\}.$$

Lumer and Rosenblum [59] credit Kleinecke for originally computing the spectrum of $\tau_{A,B}$ on $\mathcal{L}(X)$ and a proof of (2.2) can be found in the survey [77, Theorem 3.12].

We recall (M1) and (M2) imply a natural correspondence between the operators A and B^* satisfying the Hypercyclicity Criterion and, respectively, the induced multipliers L_A and R_B being hypercyclic. The pattern continues in Sections 3 and 4 of [A], where we demonstrate families of non-hypercyclic operators that induce non-hypercyclic commutator maps.

Notably, we prove that commutator maps induced by Riesz operators are never hypercyclic on any separable Banach ideal. In particular, it follows that compact operators cannot induce a hypercyclic commutator map. We do this by using the well known spectral condition from Kitai [56], which states that every connected component of the spectrum of a hypercyclic operator intersects the unit circle (cf. [14, Theorem 1.18]).

Furthermore, we show the commutator map Δ_N , induced by a normal operator N acting on a separable Hilbert space H , cannot be supercyclic on the space C_2 of Hilbert-Schmidt operators. This is done by showing that Δ_N is itself a normal operator and then applying a result of Bourdon [27], which states that normal operators cannot be supercyclic (cf. [50, Theorem 5.30]). We note Kitai [56] had previously shown that normal operators cannot be hypercyclic.

Observe that the results mentioned thus far have preserved the connection between the hypercyclicity (or non-hypercyclicity) of the operator A and the induced maps L_A and Δ_A . So one might expect that a reasonable candidate for a hypercyclic commutator map Δ_A on $\mathcal{K}(\ell^2)$ would arise from an operator A which satisfies the Hypercyclicity Criterion and induces a commutator map having (at least) dense range.

The map $\Delta_B: \mathcal{K}(\ell^2) \rightarrow \mathcal{K}(\ell^2)$ is a classical example of this kind, since scalar multiples of the backward shift cB satisfy the Hypercyclicity Criterion on ℓ^2 for $|c| > 1$ and it is well known that

$$\overline{\text{ran}}(\Delta_B) = \mathcal{K}(\ell^2).$$

However, the main result in [A] demonstrates that Δ_{cB} is not hypercyclic, for any scalar c , on any separable Banach ideal of $\mathcal{L}(X)$. In particular, our argument explicitly demonstrates that Δ_{cB} does not have a dense orbit in $\mathcal{K}(\ell^2)$. The statement of the theorem is as follows.

THEOREM 2.1. *Let $B \in \mathcal{L}(\ell^2)$ be the backward shift operator. Then the commutator map Δ_{cB} is not hypercyclic $\mathcal{K}(\ell^2) \rightarrow \mathcal{K}(\ell^2)$ for any constant c .*

The argument from Theorem 2.1 can be extended to any analytic polynomial in the backward shift $p(B)$. That is, for any polynomial $p(z) = \sum_{j=0}^m c_j z^j$, where $m \in \mathbb{N}$ and $c_j \in \mathbb{C}$, for $0 \leq j \leq m$, we define the associated operator

$$p(B) = \sum_{j=0}^m c_j B^j \in \mathcal{L}(\ell^2).$$

This gives the following generalisation of Theorem 2.1.

THEOREM 2.2. *Let $p(B): \ell^2 \rightarrow \ell^2$ be any analytic polynomial in the backward shift B . Then the induced commutator map $\Delta_{p(B)}$ is not hypercyclic on $\mathcal{K}(\ell^2)$.*

By using the hypercyclic comparison principle, we show that Theorems 2.1 and 2.2 also hold on any Banach ideal contained in $\mathcal{K}(\ell^2)$. Moreover, by essentially the same argument these results can be extended to separable Banach ideals $J \subset \mathcal{K}(X)$, where $X = \ell^p$ for $1 < p < \infty$ or $X = c_0$.

However, the argument from Theorems 2.1 and 2.2 is an ad hoc approach specific to the map Δ_{cB} . So it cannot be applied to generalise our results to arbitrary bounded linear operators.

On the other hand, considering (M1) and (M2) it is natural to wonder whether a hypercyclic commutator map could be induced by a dual hypercyclic operator. A hypercyclic operator T is said to be *dual hypercyclic* if its adjoint T^* is also hypercyclic. Herrero [54] originally posed the question whether such operators exist and examples were subsequently obtained by Salas [78, 79] and Petersson [66]. Moreover, according to Curto [31, p. 5] elementary operators ‘are inextricably connected to the properties of the left and right multipliers’, which provides further motivation for considering dual hypercyclic operators.

However, in [A] we use a spectral argument to show that the dual hypercyclic operator uncovered in [79] cannot induce a hypercyclic commutator map. The argument does not, however, apply to the dual hypercyclic operators from [78] and [66].

2.2. Hypercyclicity of Generalised Derivations

Following the investigation of the dynamics of commutator maps, in [B] we tackle the next natural question which relates to the hypercyclicity of generalised derivations

$$\tau_{A,B} = L_A - R_B.$$

We have more freedom in this case since we are dealing with pairs (A, B) of operators. Indeed, in contrast to the unresolved question of the existence of

a hypercyclic commutator map, we are able to uncover concrete classes of hypercyclic generalised derivations acting on separable Banach ideals.

We first note that obvious instances $\tau_{A,B}$ of hypercyclic generalised derivations are obtained by taking either $A \equiv 0$ or $B \equiv 0$. This reverts back to the case of the basic multipliers L_A and R_B , which was fully characterised in [21].

The type of hypercyclic generalised derivations we analyse in [B] are induced by the class of extended backward shifts. Following the terminology of [50, p. 219], we recall that $T \in \mathcal{L}(X)$ is an *extended backward shift* if

$$\text{span} \left(\bigcup_{j=0}^{\infty} (\ker T^j \cap \text{ran } T^j) \right)$$

is dense in the Banach space X .

By an unpublished result of Grivaux and Shkarin [47], which can be found in [50, Theorem 8.6], if $T \in \mathcal{L}(X)$ is an extended backward shift then the operators $I + T$ and e^T satisfy the Hypercyclicity Criterion on X . Here the operator $e^T \in \mathcal{L}(X)$ is defined as

$$e^T = \sum_{j=0}^{\infty} \frac{1}{j!} T^j$$

for any $T \in \mathcal{L}(X)$.

For brevity, we say a Banach ideal J is *admissible* when it contains the finite rank operators as a dense subset with respect to the ideal norm $\|\cdot\|_J$, that is

$$\overline{\mathcal{F}(X)}^{\|\cdot\|_J} = J.$$

Using a similar approach to that taken in [21], we show that $I + T$ and e^T give hypercyclic generalised derivations which satisfy the Hypercyclicity Criterion and are hence hypercyclic. The precise statement is as follows.

THEOREM 2.3. *Let X be a Banach space and let $J \subset \mathcal{L}(X)$ be a separable, admissible Banach ideal. If $T \in \mathcal{L}(X)$ is an extended backward shift then the generalised derivations $L_T - R_{-I}$ and $L_{T'} - R_{-I}$ satisfy the Hypercyclicity Criterion on J , where $T' = \sum_{j=1}^{\infty} \frac{1}{j!} T^j$.*

We also obtain in [B] many families of non-hypercyclic generalised derivations and we extend some observations from [A] on commutator maps to the generalised derivation case.

In particular, for a Banach space X and a separable Banach ideal $J \subset \mathcal{L}(X)$, we show if $A, B \in \mathcal{L}(X)$ are Riesz operators then the induced generalised derivation $\tau_{A,B}: J \rightarrow J$ is not hypercyclic. This is done by applying a similar spectral argument to that used in [A].

For a Hilbert space H , if $A, B \in \mathcal{L}(H)$ are such that A and B^* are hyponormal, then we also show that the generalised derivation $\tau_{A,B}$ is not supercyclic on the space C_2 of Hilbert-Schmidt operators. We recall that $U \in \mathcal{L}(H)$ is *positive* if for all $x \in H$ the inner product

$$\langle Ux, x \rangle \geq 0$$

and that $U \in \mathcal{L}(H)$ is *hyponormal* if $U^*U - UU^*$ is positive. The hyponormal operators contain some well known classes of operators such as

the subnormal, normal and self-adjoint operators [53]. Our argument uses a result of Bourdon [27], which states that hyponormal operators cannot be supercyclic.

We also identify a necessary spectral condition for the hypercyclicity of generalised derivations. For $A, B \in \mathcal{L}(X)$, if both point spectra

$$(2.3) \quad \sigma_p(A^*) \neq \emptyset \quad \text{and} \quad \sigma_p(B) \neq \emptyset$$

then it follows that the generalised derivation $\tau_{A,B}$ is not hypercyclic on any separable Banach ideal $J \subset \mathcal{L}(X)$.

In fact, this spectral condition can be generalised to a particular family of elementary operators. The argument uses the well known fact that the adjoint of a hypercyclic operator cannot possess any eigenvalues (cf. [14, Proposition 1.17]).

PROPOSITION 2.4. *Let X be a Banach space and $A = (A_j)_{j=1}^n, B = (B_j)_{j=1}^n \in \mathcal{L}(X)^n$ for $n \geq 1$. If the operators A_j^* have eigenvalues sharing a common eigenvector and the operators B_j have eigenvalues sharing a common eigenvector, for $1 \leq j \leq n$, then the elementary operator $\mathcal{E}_{A,B}$ is not hypercyclic on any separable Banach ideal $J \subset \mathcal{L}(X)$.*

2.3. Hypercyclicity on the Argyros-Haydon Space

Argyros and Haydon [7] resolved the famous *scalar-plus-compact* problem with the construction of the extreme Banach space X_{AH} . While this type of space is rare, X_{AH} is relatively nice and possesses many remarkable properties. In [B] we reveal some surprising differences between the hypercyclic behaviour of particular classes of elementary operators acting on separable Banach ideals of $\mathcal{L}(X_{AH})$ and on $\mathcal{L}(X_{AH})$ itself.

We first recall that the space X_{AH} has a Schauder basis and every $T \in \mathcal{L}(X_{AH})$ is of the form

$$(2.4) \quad T = \lambda I + K$$

where $\lambda \in \mathbb{C}$ and $K \in \mathcal{K}(X_{AH})$ is a compact operator.

The existence of a Schauder basis implies that X_{AH} possesses the approximation property and it is shown in [7] that the dual X_{AH}^* is isomorphic to the sequence space ℓ^1 and is therefore separable. Hence the space of compact operators $\mathcal{K}(X_{AH})$ is a separable, admissible Banach ideal under the operator norm topology and it further follows that $\mathcal{L}(X_{AH}) = \mathbb{C} \cdot I + \mathcal{K}(X_{AH})$ is separable in the operator norm topology.

The separability of $\mathcal{L}(X_{AH})$ naturally leads to the question of whether it supports hypercyclic elementary operators. In [50, p. 300] an argument is partially outlined showing that the left multiplier is not hypercyclic on $\mathcal{L}(X_{AH})$. In [B] we establish a somewhat more general observation for elementary operators acting on $\mathcal{L}(X_{AH})$.

We initially state our findings in the general setting of Banach algebras. For a Banach algebra \mathcal{A} , the elementary operator $\mathcal{E}_{a,b}: \mathcal{A} \rightarrow \mathcal{A}$ is given by

$$\mathcal{E}_{a,b} = \sum_{j=1}^n L_{a_j} R_{b_j}$$

where $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n) \in \mathcal{A}^n$, $n \geq 1$ and we define $L_{a_j}(s) = a_j s$, $R_{b_j}(s) = s b_j$ for any $s \in \mathcal{A}$ and $j = 1, \dots, n$.

We further recall that the nonzero linear functional $\varphi: \mathcal{A} \rightarrow \mathbb{C}$ is said to be a non-trivial *multiplicative linear functional* if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{A}$. It is well known that they are continuous.

The theorem statement is as follows.

THEOREM 2.5. *Let \mathcal{A} be a Banach algebra which admits a non-trivial multiplicative linear functional $\varphi \in \mathcal{A}^*$. Then the elementary operator $\mathcal{E}_{a,b}: \mathcal{A} \rightarrow \mathcal{A}$ is not hypercyclic.*

We then identify in [B] a non-trivial linear multiplicative functional on the Banach algebra $\mathcal{L}(X_{AH})$, so it follows from Theorem 2.5 that no elementary operator is hypercyclic on $\mathcal{L}(X_{AH})$.

On the other hand, Theorem 2.3 gives instances of generalised derivations $L_T - R_{-I}$, that are hypercyclic on $\mathcal{H}(X_{AH})$. So in light of Theorems 2.3 and 2.5, the seemingly minor change of one dimension completely alters the hypercyclicity property of the generalised derivation $L_T - R_{-I}$ when acting on $\mathcal{L}(X_{AH})$.

Furthermore, in [B] we also prove the surprising result that commutator maps are never hypercyclic on any Banach ideal of $\mathcal{L}(X_{AH})$.

2.4. Concluding Remarks

Our investigation has revealed something of the subtle nature of the hypercyclicity of classes of elementary operators.

In [A] we initiated the investigation of the hypercyclicity properties of commutator maps and in [B] we identified particular Banach ideals which do not admit any hypercyclic commutator maps. However, in contrast to the commutator maps, instances of hypercyclic generalised derivations were shown to exist in [B]. This gives rise to the following natural questions.

1. Does there exist a separable Banach ideal $J \subset \mathcal{L}(X)$ and an operator $A \in \mathcal{L}(X)$ such that the commutator map $\Delta_A: J \rightarrow J$ is hypercyclic?
2. Are the dual hypercyclic operators from [78] or [66] suitable candidates to induce a hypercyclic commutator map?
3. Do reasonable sufficient conditions exist on the pair (A, B) that give hypercyclic generalised derivations $\tau_{A,B}$ on separable Banach ideals?
4. Can we identify instances of hypercyclic generalised derivations $\tau_{A,B}$ where both A and B are different from the identity operator?

3. Growth Rates of Frequently Hypercyclic Harmonic Functions

In [C] we identify minimal L^2 -growth rates of the harmonic functions that are frequently hypercyclic for the partial differentiation operator

$$\frac{\partial}{\partial x_j}: \mathcal{H}(\mathbb{R}^N) \rightarrow \mathcal{H}(\mathbb{R}^N)$$

acting on the space $\mathcal{H}(\mathbb{R}^N)$ of harmonic functions on \mathbb{R}^N , where $N \geq 2$ and $1 \leq j \leq N$. This answers a question originally posed by Blasco et al. [19].

We recall a twice continuously differentiable function $h: \mathbb{R}^N \rightarrow \mathbb{R}$ is *harmonic* if it is a solution to the Laplace equation, that is

$$\Delta h \equiv 0$$

where Δ denotes the *Laplacian* given by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_N^2}.$$

The study of harmonic functions, known as potential theory, has its origins in Newtonian physics and it is connected to classical questions such as the heat equation and the Dirichlet problem. More recently it finds applications in probability theory and the study of Markov chains [36, p. xxi]. Comprehensive introductions to the area can be found in [9] and [8].

3.1. Growth of Frequently Hypercyclic Functions

We denote by $S(r)$ the sphere of radius r centred at the origin of \mathbb{R}^N . We consider the growth rates of harmonic functions $h \in \mathcal{H}(\mathbb{R}^N)$ in terms of the average L^2 -norm on $S(r)$ given by

$$M_2(h, r) = \left(\int_{S(r)} |h|^2 d\sigma_r \right)^{1/2}$$

where $r > 0$ and σ_r is the normalised $(N - 1)$ -dimensional surface measure on $S(r)$ with $\sigma_r(S(r)) = 1$. $M_2(h, \cdot)$ is also sometimes referred to as the 2-integral mean of h .

The space $\mathcal{H}(\mathbb{R}^N)$ is a Fréchet space with the complete metric

$$(3.1) \quad d(g, h) = \sum_{n=1}^{\infty} 2^{-n} \frac{\sup_{|x|=n} |g(x) - f(x)|}{1 + \sup_{|x|=n} |g(x) - f(x)|}$$

for $g, h \in \mathcal{H}(\mathbb{R}^N)$ and the d -metric topology is the topology of local uniform convergence.

We recall for the basic differentiation operator

$$D: f \mapsto f'$$

acting on the space $H(\mathbb{C})$ of entire functions, that MacLane [60] constructed a universal entire function $f \in H(\mathbb{C})$ such that the sequence of derivatives (f, f', f'', \dots) is dense in $H(\mathbb{C})$. Moreover, initial growth rates for universal

entire functions f were already given in [60]. Duños-Ruis [38] improved the growth estimates and sharp growth rates were identified independently by Grosse-Erdmann [48] and Shkarin [81].

Recall the space $H(\mathbb{C})$ is a Fréchet space with the same complete metric as (3.1) and it is separable when endowed with the topology of local uniform convergence. The growth rates for $f \in H(\mathbb{C})$ are in terms of the sup-norm on spheres of radius $r > 0$, which is given by

$$M_\infty(f, r) = \sup_{|z|=r} |f(z)|$$

and also in terms of the average L^p -norms on spheres of radius $r > 0$

$$M_p(f, r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p}$$

for $1 \leq p < \infty$.

The differentiation operator D was shown to be frequently hypercyclic on $H(\mathbb{C})$ by Bayart and Grivaux [12] and the question of growth rates of frequently hypercyclic entire functions was raised in [25]. Initial growth estimates were given by Blasco et al. [19] and Bonet and Bonilla [20]. The growth rates in [19] were obtained by applying a generalisation of the Frequent Hypercyclicity Criterion from [24] and in [20] they used an eigenvalue criterion from [46]. The setting for both [19] and [20] was that of a separable weighted Banach space which is densely embedded in $(H(\mathbb{C}), d)$.

To obtain minimal growth rates, Drasin and Saksman [37] explicitly constructed a suitable D -frequently hypercyclic entire function. In the case $1 < p \leq \infty$, they proved for all $c > 0$ that there exists an entire D -frequently hypercyclic function $f \in H(\mathbb{C})$ with

$$(3.2) \quad M_p(f, r) \leq c \frac{e^r}{r^{a(p)}}$$

for all $r > 0$ and where

$$a(p) = \max \left(\frac{1}{4}, \frac{1}{2p} \right).$$

For $p = 1$, they showed for any given function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, such that $\varphi(r) \rightarrow \infty$ as $r \rightarrow \infty$, there exists an entire D -frequently hypercyclic function $f \in H(\mathbb{C})$ with

$$M_1(f, r) \leq \varphi(r) \frac{e^r}{r^{1/2}}$$

for all $r > 0$. This also gives that the growth rates obtained in [20] are sharp in the case $p = 1$.

Returning to the partial differentiation operator $\partial/\partial x_j$ acting on $\mathcal{H}(\mathbb{R}^N)$, Aldred and Armitage [1] proved for any given function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, with $\varphi(r) \rightarrow \infty$ as $r \rightarrow \infty$, that there exists a $\partial/\partial x_j$ -hypercyclic harmonic function $h \in \mathcal{H}(\mathbb{R}^N)$ such that

$$M_2(h, r) \leq \varphi(r) \frac{e^r}{r^{(N-1)/2}}$$

for $r > 0$. They also showed there does not exist a $\partial/\partial x_j$ -hypercyclic $h \in \mathcal{H}(\mathbb{R}^N)$ that satisfies

$$M_2(h, r) \leq C \frac{e^r}{r^{(N-1)/2}}$$

for $r > 0$ and where $C > 0$ is constant. We note their results are for the more general notion of universality.

In the frequently hypercyclic case, Blasco et al. [19] identified the following growth rates.

1. Let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be any function with $\varphi(r) \rightarrow \infty$ as $r \rightarrow \infty$. Then there exists a $\partial/\partial x_j$ -frequently hypercyclic $h \in \mathcal{H}(\mathbb{R}^N)$ with

$$(3.3) \quad M_2(h, r) \leq \varphi(r) \frac{e^r}{r^{N/2-3/4}}$$

for $r > 0$ sufficiently large.

2. Let $\psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be any function with $\psi(r) \rightarrow 0$ as $r \rightarrow \infty$. Then there is no $\partial/\partial x_j$ -frequently hypercyclic $h \in \mathcal{H}(\mathbb{R}^N)$ with

$$M_2(h, r) \leq \psi(r) \frac{e^r}{r^{N/2-3/4}}$$

for $r > 0$ sufficiently large.

To obtain the above growth rates in [19], they again applied a generalisation of the Frequent Hypercyclicity Criterion in the setting of a separable weighted Banach space which is densely embedded in $(\mathcal{H}(\mathbb{R}^N), d)$.

They also asked [19, Section 6] whether there exists a $\partial/\partial x_j$ -frequently hypercyclic $h \in \mathcal{H}(\mathbb{R}^N)$ such that the function φ in (3.3) can be replaced with a constant. In [C] we answer this question in the positive by explicitly constructing a suitable $\partial/\partial x_j$ -frequently hypercyclic harmonic function. This is done by modifying the approach of Drasin and Saksman from [37] and the statement of the theorem is as follows.

THEOREM 3.1. *Let $N \geq 2$ and $1 \leq j \leq N$. Then for any constant $C > 0$ there exists a $\partial/\partial x_j$ -frequently hypercyclic harmonic function $h \in \mathcal{H}(\mathbb{R}^N)$ such that*

$$(3.4) \quad M_2(h, r) \leq C \frac{e^r}{r^{N/2-3/4}}$$

for all $r > 0$.

Since the real part of an entire function is a harmonic function, the case $N = 2$ in Theorem 3.1 can be deduced from (3.2) when $p = 2$. So the argument of Theorem 3.1 is essentially concerned with the case $N \geq 3$. Moreover, since potential theory in the plane differs from that in higher dimensions, it turns out the proof of Theorem 3.1 for $N \geq 3$ is significantly more involved than in the entire case.

3.2. The Harmonic Function h

The definition of the harmonic function h satisfying Theorem 3.1 is quite technical, so in this section we briefly present some intuition into the key aspects of its structure and growth.

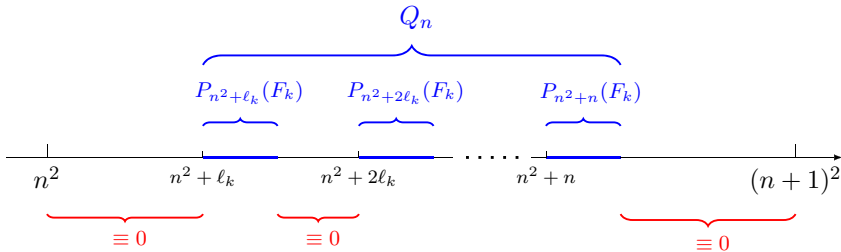


FIGURE 3.1. Degrees of the polynomials contained in Q_n

To iterate $h \in \mathcal{H}(\mathbb{R}^N)$ under the partial differentiation operator $\partial/\partial x_j$, for $j \in \{1, \dots, N\}$, we of course require some candidate for an antiderivative. The appropriate antiderivate was defined by Aldred and Armitage [1], using a specific orthogonal representation of harmonic polynomials which was constructed by Kuran [57]. In particular, for every harmonic polynomial H we have for each $n \in \mathbb{N}$ a linear map

$$P_n: H \mapsto P_n(H),$$

called the n^{th} *primitive* of H , such that $P_n(H)$ is a harmonic polynomial and

$$\frac{\partial^n}{\partial x_j^n} P_n(H) = H.$$

The required function h is defined as the sum of particular harmonic polynomials Q_n , for $n \in \mathbb{N}$, that is

$$h = \sum_{n=0}^{\infty} Q_n.$$

Moreover, the nonzero Q_n are composed of primitives of suitable harmonic polynomials.

To give an idea of the structure of each Q_n , we begin by setting

$$Q_n \equiv 0$$

when n is odd and $n = 0$.

Next we consider a sequence $(F_k)_{k \geq 1}$ of harmonic polynomials, which is dense in $\mathcal{H}(\mathbb{R}^N)$ and for each $k \geq 1$ we associate a number $\ell_k \in \mathbb{N}$. The choice of the ℓ_k turns out to be vital in satisfying the requirements of Theorem 3.1.

For each even $n \in \mathbb{N}$, which contributes a nonzero Q_n , we associate a particular F_k from our dense sequence. These n are chosen suitably large and we define

$$Q_n = P_{n^2+\ell_k}(F_k) + P_{n^2+2\ell_k}(F_k) + \dots + P_{n^2+n}(F_k).$$

Thus Q_n is composed of primitives of the associated harmonic polynomial F_k . We note that the degrees of the polynomials contained in Q_n are supported on the interval $(n^2, (n+1)^2)$, as illustrated in Figure 3.1.

The crucial ideas in the construction of h are that we consider $n \in \mathbb{N}$ sufficiently large and that we are free to choose the numbers in the sequence

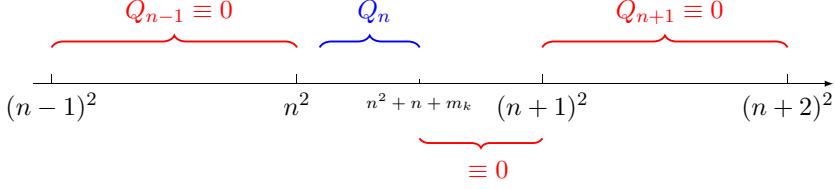


FIGURE 3.2. Degrees of the polynomials Q_n ($m_k = \deg F_k$)

(ℓ_k) as large as required. This gives that the primitives contained in the Q_n are disjointly supported, as illustrated in Figure 3.1, and that the nonzero Q_n are disjointly supported, as can be seen in Figure 3.2. This is critical in proving both the frequent hypercyclicity and the prescribed growth of h .

Finally, we describe very briefly the calculation of the growth in the L^2 -norm of h on spheres of radius $r > 0$. Following some technicalities, the required estimates boil down to identifying suitable upper bounds for sums of the form

$$(3.5) \quad \sum_{j=2\ell_k}^{\infty} \frac{r^{2j\ell_k}}{(j\ell_k)!^2 (j\ell_k + 1)^{N-2}}$$

for each $k \geq 1$. To illustrate (3.5), we fix $r > 0$ and $N \geq 2$ and consider the function

$$p(x) = \frac{r^{2x}}{x!^2 (x+1)^{N-2}}$$

for $x \in [0, \infty)$. Notice that $p(x)$ achieves its maximum close to the point $x = r$ and it then rapidly decays, as illustrated in Figure 3.3.

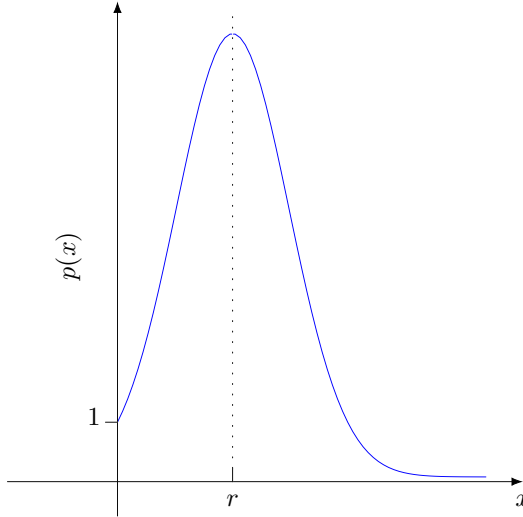


FIGURE 3.3. Growth of $p(x)$

Furthermore, a result of Barnes [10] gives that there exists a constant $C > 0$ such that for all $r > 0$

$$(3.6) \quad \sum_{j=0}^{\infty} \frac{r^{2j}}{j!^2(j+1)^{N-2}} < C \frac{e^{2r}}{r^{N-3/2}}.$$

We note that (3.6) was used to obtain the growth estimates in [19].

However, in (3.5) we are not summing over all natural numbers since the sum contains gaps of length ℓ_k and an improved estimate is given in Lemma 6.1 of [C]. The prescribed growth rate of Theorem 3.1 then follows by choosing the integers in the sequence (ℓ_k) sufficiently large.

3.3. Concluding Remarks

The identification of minimal L^2 -growth of frequently hypercyclic harmonic functions gives rise to the following fundamental questions.

1. Is it possible to obtain the analogue of Theorem 3.1 in the L^p case for $p \neq 2$? In particular, the treatment of the L^p -norms in [37] relied on three key ingredients, namely using Rudin-Shapiro polynomials, heat kernel estimates and the idea of constructing polynomial blocks which are independently supported on the intervals $[n^2, (n+1)^2)$, for $n \geq 1$. We employed the blocking technique in [C], but it is unclear if there exist suitable analogues of the first two ingredients in this setting.
2. The general partial differentiation operators $D^\alpha: \mathcal{H}(\mathbb{R}^N) \rightarrow \mathcal{H}(\mathbb{R}^N)$ are defined by

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$$

where $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ and $|\alpha| = \alpha_1 + \cdots + \alpha_N$. In [2] and [19] growth rates were also given for D^α -frequently hypercyclic harmonic functions. These growth rates are not sharp, so can we find minimal growth rates for D^α in terms of L^p -norms for $1 \leq p \leq \infty$? It is also natural to consider linear combinations of the more general differentiation operators.

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